

Gravitational Actions in Two Dimensions and the Mabuchi Functional

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The Mabuchi energy is an interesting geometric functional on the space of Kähler metrics that plays a crucial rôle in the study of the geometry of Kähler manifolds. We show that this functional, as well as other related geometric actions, contribute to the effective gravitational action when a massive scalar field is coupled to gravity in two dimensions in a small mass expansion. This yields new theories of two-dimensional quantum gravity generalizing the standard Liouville models.

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1 Introduction

When a matter quantum field theory is coupled to gravity, the metric dependence of the partition function induces an effective gravitational action. On a two-dimensional euclidean space-time, when the quantum field theory is conformally invariant, this effective action is always proportional to the Liouville action [1]. In the so-called conformal gauge, for which the metric g is proportional to a fixed background metric g_0 ,

$$g = e^{2\sigma} g_0, \quad (1.1)$$

the Liouville action takes the simple form

$$S_L(g_0, g) = \int d^2x \sqrt{g_0} (g_0^{ab} \partial_a \sigma \partial_b \sigma + R_0 \sigma), \quad (1.2)$$

where R_0 is the Ricci scalar for the metric g_0 . The resulting Liouville models for two-dimensional quantum gravity have been extensively studied in the literature.

The gravitational coupling of non-conformally invariant matter has been much less studied (see e.g. [2]). In this case, one expects a complicated and model-dependent gravitational action. It is unclear a priori if new interesting and simple actions, yielding models with nice geometrical and physical features, can emerge.

Recently, the authors have explained that two-dimensional quantum gravity could be naturally formulated in the context of Kähler geometry [3, 4]. In this context, mathematicians have much used and studied natural functionals defined on the space of Kähler potentials, the most salient being the Mabuchi action [5], also called the Mabuchi energy in the mathematical literature. Our aim in the present work is to investigate whether such actions could contribute to the gravitational effective action in some simple physical models. We shall see that they do, yielding new models of two-dimensional quantum gravity with potentially very rich mathematical and physical features.

Our space-time will always be a compact Riemann surface Σ_h of genus h . An arbitrary riemannian metric on Σ_h can be parametrized by a finite set of complex moduli τ and the conformal factor $e^{2\sigma}$, with a background metric $g_0(\tau)$ in (1.1) of the form $g_0 = 2g_{0z\bar{z}}|dz|^2$. In the Kähler framework, we write the conformal factor $e^{2\sigma}$ in terms of the area A (which is the unique Kähler modulus in two dimensions) and the Kähler potential ϕ ,

$$e^{2\sigma} = \frac{A}{A_0} - \frac{1}{2} A \Delta_0 \phi, \quad (1.3)$$

where A_0 and Δ_0 are the area and the positive laplacian for the metric g_0 respectively. The $\partial\bar{\partial}$ lemma shows that the differential equation (1.3) can always be solved for A and ϕ in terms of σ , and the solution is unique up to constant shifts in ϕ .

There are many reasons to use the Kähler potential ϕ in the context of two-dimensional quantum gravity. In particular it is possible to rigorously define in terms

of ϕ a regularized version of the gravitational path integrals [3, 4]. For our present purposes, the variable ϕ is useful because interesting functionals of the metric are most naturally written down in terms of ϕ . A famous example is the Mabuchi action [5],

$$S_M(g_0, g) = \int_{\Sigma_h} d^2x \sqrt{g_0} \left[-2\pi(1-h)\phi\Delta_0\phi + \left(\frac{8\pi(1-h)}{A_0} - R_0 \right) \phi + \frac{4}{A} \sigma e^{2\sigma} \right]. \quad (1.4)$$

This action and its higher dimensional generalizations play a central rôle in Kähler geometry. It is well defined on the space of metrics, because it is unchanged if ϕ is shifted by a constant. It is bounded from below and convex, which makes it a good candidate for an action to be used in a path integral. It satisfies the so-called cocycle conditions,

$$S_M(g_1, g_2) = -S_M(g_2, g_1), \quad (1.5)$$

$$S_M(g_1, g_3) = S_M(g_1, g_2) + S_M(g_2, g_3), \quad (1.6)$$

which can be checked straightforwardly from the definition (1.4). The same cocycle identities are also satisfied by the Liouville action. As we shall review below, these identities actually are fundamental consistency conditions that any effective gravitational action must satisfy. Finally, the critical points of the Mabuchi action are the metrics of constant scalar curvature, another property shared in two dimensions with the Liouville action. This property is also valid for the higher dimensional generalizations of (1.4), which goes a long way in explaining the central rôle played by the Mabuchi action in the study of such metrics on general Kähler manifolds. For more details and references on the profound geometrical properties of the space of metrics on a Kähler manifold, we refer the reader to [4, 6, 7] and the comprehensive recent review [8].

We focus in the following on the simple model of a massive scalar field X with action

$$S_{\text{mat}}(X, g; q, m) = \frac{1}{8\pi} \int d^2x \sqrt{g} (g^{ab} \partial_a X \partial_b X + qRX + m^2 X^2). \quad (1.7)$$

Our main result is to show that the Mabuchi action and other simple functionals of the Kähler potential ϕ , like the so-called Aubin-Yau action, contribute to the gravitational effective action for this model in a small mass expansion. We also study the gravitational dressing of operators of the form

$$\mathcal{O}_k = e^{kX} \quad (1.8)$$

and show that it involves the Aubin-Yau action on top of the familiar dressing factors found in the conformal field theory limit.

A striking feature is that it is always possible to cancel out the familiar Liouville term in the effective action, for example by coupling with a suitable spectator conformal field theory. The resulting pure Mabuchi theories are entirely new and intriguing two-dimensional quantum gravity models.

Plan of the paper

We start in Section 2 with a brief review on two-dimensional quantum gravity, gravitational effective actions and gravitational dressing. In Section 3, we discuss the basic properties of various actions, in particular Mabuchi's and Aubin-Yau's. We derive fundamental identities relating these actions to the variations of various functionals associated with the Laplace operator. In Section 4, we apply the results of Section 3 to compute the gravitational effective action and the gravitational dressing of the operators e^{kX} for the model (1.7). We also compute the trace of the stress-energy tensor. Finally, in Section 5, we conclude and discuss possible extensions of our work. We have tried to make the presentation as elementary and self-contained as possible. In particular we have included two Appendices containing simple derivations of results used in the main text.

Notations

To a metric $g = 2g_{z\bar{z}}|dz|^2$ on Σ_h we associate its Kähler form,

$$\omega = ig_{z\bar{z}}dz \wedge d\bar{z}. \quad (1.9)$$

In two dimensions, the Kähler form coincides with the volume form and thus in particular the area is given by

$$A(g) = \int_{\Sigma_h} \omega. \quad (1.10)$$

The positive laplacian for the metric g , whose determinant we also denote by g , is defined as usual by

$$\Delta f = -\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b f). \quad (1.11)$$

In terms of the standard Dolbeault operators ∂ and $\bar{\partial}$ we have

$$\partial\bar{\partial}f = \frac{i}{2}\Delta f \omega, \quad (1.12)$$

from which it can be seen that the relation (1.3) is equivalent to

$$\omega = \frac{A}{A_0}\omega_0 + iA\partial\bar{\partial}\phi. \quad (1.13)$$

The space of Kähler potentials ϕ is simply

$$\mathcal{H}_{\omega_0} = \{\phi : \Sigma \rightarrow \mathbb{R} \mid A_0\Delta_0\phi < 2\}, \quad (1.14)$$

the condition on ϕ ensuring the strict positivity of the metrics, $g_{z\bar{z}} > 0$. The scalar curvature, or Ricci scalar R , is such that

$$R\omega = -i\partial\bar{\partial}\ln g. \quad (1.15)$$

The integral of the scalar curvature is a topological invariant,

$$\int_{\Sigma_h} R \omega = 8\pi(1 - h) . \quad (1.16)$$

This is a useful formula, from which for instance the invariance of (1.4) under constant shifts of ϕ is derived. Finally, the laplacians Δ and Δ_0 and scalar curvatures R and R_0 of two metrics g and g_0 related by a formula of the form (1.1) are themselves linked by the simple relations

$$e^{2\sigma} \Delta = \Delta_0 , \quad (1.17)$$

$$e^{2\sigma} R = R_0 + 2\Delta_0 \sigma . \quad (1.18)$$

2 On gravitational effective actions and dressing

2.1 Basic definitions

2.1.1 Gravitational effective actions

The partition function Z_{mat} of a matter quantum field theory, defined on the two-dimensional euclidean space-time Σ_h endowed with a fixed metric g , with fields X , couplings λ and action $S_{\text{mat}}(X; \lambda; g)$, is defined by the QFT path integral

$$Z_{\text{mat}}(\lambda; g) = \int \mathcal{D}X e^{-S_{\text{mat}}(X; \lambda; g)} . \quad (2.1)$$

The quantum gravity partition function Z is obtained by integrating further over the space of metrics on Σ_h ,

$$Z(\mu, \lambda) = \int \mathcal{D}g Z_{\text{mat}}(\lambda; g) e^{-\mu A(g)} , \quad (2.2)$$

where μ is the bare cosmological constant. It is also interesting to consider the partition function Z_A at fixed area, such that

$$Z_A(\lambda) = \int \mathcal{D}g Z_{\text{mat}}(\lambda; g) \delta\left(\int_{\Sigma_h} d^2x \sqrt{g} - A\right) , \quad (2.3)$$

$$Z(\mu, \lambda) = \int dA e^{-\mu A} Z_A(\lambda) . \quad (2.4)$$

The above formal path integrals over metrics are of course infinite because of diffeomorphism invariance and the fact that the diffeomorphism group has infinite volume. As is well-known, the effect of fixing the gauge by imposing (1.1) is to replace

$$\mathcal{D}g \rightarrow \mathcal{D}\sigma Z_{\text{gh}}(g) , \quad (2.5)$$

where the volume form $\mathcal{D}\sigma$ is associated with the curved riemannian metric

$$\|\delta\sigma\|^2 = \int_{\Sigma} d^2x \sqrt{g} (\delta\sigma)^2 = \int_{\Sigma} d^2x \sqrt{g_0} e^{2\sigma} (\delta\sigma)^2 \quad (2.6)$$

on the space of conformal factors and $Z_{\text{gh}}(g)$ is the partition function of a ghost conformal field theory of central charge $c_{\text{gh}} = -26$ corresponding to the Faddeev-Popov determinant. We end up with

$$Z(\mu, \lambda) = \int \mathcal{D}\sigma e^{-\mu A(g)} Z_{\text{mat+gh}}(\lambda; g), \quad (2.7)$$

where

$$Z_{\text{mat+gh}}(\lambda; g) = Z_{\text{mat}}(\lambda; g) \times Z_{\text{gh}}(g) \quad (2.8)$$

the partition function of the matter plus ghost QFT. A regularization of the path integral over σ and in particular a rigorous construction of the volume form $\mathcal{D}\sigma$ is given in [3, 4].

The *gravitational effective action*, or simply the gravitational action, is defined by

$$S_{\text{grav}}(g_0, g; \lambda) = -\ln \frac{Z_{\text{mat+gh}}(\lambda, g)}{Z_{\text{mat+gh}}(\lambda, g_0)}. \quad (2.9)$$

It depends on the metric g , on a reference (or background) metric g_0 and on the couplings λ in the matter QFT. In terms of S_{grav} the partition function reads

$$Z(\mu, \lambda) = Z_{\text{mat+gh}}(\lambda; g_0) \int \mathcal{D}\sigma e^{-\mu A(g) - S_{\text{grav}}(g_0, g; \lambda)} \quad (2.10)$$

$$= Z_{\text{mat+gh}}(\lambda; g_0) \times Z_{\text{grav}}(\mu, \lambda; g_0) \quad (2.11)$$

where the gravitational partition function is defined by

$$Z_{\text{grav}}(\mu, \lambda; g_0) = \int \mathcal{D}\sigma e^{-\mu A(g) - S_{\text{grav}}(g_0, g; \lambda)}. \quad (2.12)$$

The calculation of the quantum gravity partition function can thus be split into two parts. The first part is a standard QFT calculation in a *fixed* background metric yielding $Z_{\text{mat+gh}}$. The second part is a gravitational partition function governed by the gravitational action S_{grav} .

2.1.2 Gravitational dressing

The above discussion can be repeated straightforwardly when operators are inserted in the path integral. For example, correlators of scalar operators in the matter QFT are defined by

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{mat}, g} = \frac{1}{Z_{\text{mat}}(\lambda; g)} \int \mathcal{D}X \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S_{\text{mat}}(X; \lambda; g)}. \quad (2.13)$$

The integrated version of these correlators

$$\int_{\Sigma_h} d^2x_1 \sqrt{g(x_1)} \cdots \int_{\Sigma_h} d^2x_n \sqrt{g(x_n)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{mat}, g} \quad (2.14)$$

is diffeomorphism invariant and can be averaged over metrics, yielding the quantum gravity correlators. If we introduce the *gravitational dressing* of the QFT correlators, defined by

$$D_{\mathcal{O}_1 \cdots \mathcal{O}_n}(x_1, \dots, x_n; \lambda; g_0, g) = e^{2 \sum_{i=1}^n \sigma(x_i)} \frac{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{mat}, g}}{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{mat}, g_0}}, \quad (2.15)$$

then we can cast the quantum gravity correlators in the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\Sigma_h} d^2x_1 \sqrt{g_0(x_1)} \cdots \int_{\Sigma_h} d^2x_n \sqrt{g_0(x_n)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{mat}, g_0} \langle D_{\mathcal{O}_1 \cdots \mathcal{O}_n}(x_1, \dots, x_n; \lambda; g_0, g) \rangle_{\text{grav}}. \quad (2.16)$$

As for the case of the partition function, the calculation of the correlator splits into two parts, a standard QFT calculation yielding the matter QFT correlator and the calculation of a correlator

$$\langle D_{\mathcal{O}_1 \cdots \mathcal{O}_n} \rangle = \frac{1}{Z_{\text{grav}}(\mu, \lambda; g_0)} \int \mathcal{D}\sigma D_{\mathcal{O}_1 \cdots \mathcal{O}_n} e^{-\mu A(g) - S_{\text{grav}}(g_0, g; \lambda)} \quad (2.17)$$

in the effective gravitational theory governed by S_{grav} .

2.1.3 The case of a conformal field theory

In the case of a matter CFT of central charge c , the gravitational action is simply [1]

$$S_{\text{grav}}(g_0, g; c) = \frac{26 - c}{24\pi} S_L(g_0, g), \quad (2.18)$$

where S_L is the Liouville action (1.2). Note that $c - 26$ is the total central charge of the matter plus ghost system. The gravitational dressing can also be exactly determined for CFT operators. For example, if the operators \mathcal{O}_i , $1 \leq i \leq n$, are conformal primaries of conformal dimensions Δ_i (the case of a marginal operator corresponding to $\Delta = 2$), then the dressing is simply

$$D_{\mathcal{O}_1 \cdots \mathcal{O}_n}(x_1, \dots, x_n; \lambda; g_0, g) = e^{\sum_{i=1}^n (2 - \Delta_i) \sigma(x_i)}. \quad (2.19)$$

2.2 Consistency conditions

Any model (associated with a general matter QFT, not necessarily conformal) must satisfy two basic consistency requirements: background independence and the cocycle identities.

2.2.1 Background independence

Background independence is simply the statement that the quantum gravity theory does not depend on the choice of background metric g_0 in (1.1). For example, the partition function Z in (2.10) does not depend on g_0 and thus the g_0 dependence must cancel on the right-hand side of (2.11) between $Z_{\text{mat+gh}}$ and Z_{grav} . Equivalently, the total matter plus ghost plus gravitational system must be a *conformal field theory of central charge zero*. Note that this is true even if the matter QFT is not conformal. Similarly, since the left-hand side of (2.16) is independent of the reference metric g_0 , the same must be true for the right-hand side.

2.2.2 Cocycle identities

Any gravitational effective action must satisfy cocycle identities that follow immediately from the definition (2.9),

$$S_{\text{grav}}(g_1, g_2) = -S_{\text{grav}}(g_2, g_1), \quad (2.20)$$

$$S_{\text{grav}}(g_1, g_3) = S_{\text{grav}}(g_1, g_2) + S_{\text{grav}}(g_2, g_3). \quad (2.21)$$

It is instructive to check by hand that the Liouville action (1.2) and the Mabuchi action (1.4) both satisfy these identities. We shall build other functionals in the next Section that also satisfy the same constraints.

The identity (2.21) has a useful differential formulation. In general, let us consider any functional $S(g_1, g_2)$ satisfying

$$S(g_1, g_3) = S(g_1, g_2) + S(g_2, g_3). \quad (2.22)$$

Taking the functional derivative of (2.22) with respect to g_3 we get

$$\frac{\delta S(g_1, g_3)}{\delta g_3(x)} = \frac{\delta S(g_2, g_3)}{\delta g_3(x)}. \quad (2.23)$$

This being valid for any g_1 and g_2 , we conclude that

$$\frac{\delta S(g_0, g)}{\delta g_{ab}(x)} = \frac{1}{4\pi} t^{ab}(x; g) \quad (2.24)$$

depends only on g and not on g_0 (the factor $1/(4\pi)$ in (2.24) is conventional). Since we shall always be in the gauge (1.1), the variation of the metric is given by

$$\delta g_{ab} = 2g_{ab}\delta\sigma \quad (2.25)$$

and the variation of S can thus be written

$$\delta_g S(g_0, g) = \frac{1}{2\pi} \int_{\Sigma_h} d^2x \sqrt{g(x)} t(x; g) \delta\sigma(x), \quad (2.26)$$

in terms of the trace $t = g_{ab}t^{ab} = t^a_a$ that depends on g but not on g_0 . Equivalently, from (1.3) we see that a variation of σ is equivalent to a variation of A and ϕ related by

$$\delta\sigma = \frac{\delta A}{2A} - \frac{1}{4}A\Delta\delta\phi, \quad (2.27)$$

and thus equation (2.26) is equivalent to

$$\delta_g S(g_0, g) = \frac{\delta A}{4\pi A} \int_{\Sigma_h} d^2x \sqrt{g} t - \frac{A}{8\pi} \int_{\Sigma_h} d^2x \sqrt{g} \Delta t \delta\phi. \quad (2.28)$$

Conversely, if any of the equivalent conditions (2.24), (2.26) or (2.28) is satisfied, then

$$S(g_0, g) = \Psi(g) - \Psi(g_0) \quad (2.29)$$

for a primitive Ψ of t with respect to g . Such a functional S automatically satisfies (2.21) and $S(g_0, g) = -S(g, g_0)$ as well.

When S is not an arbitrary abstract functional satisfying (2.22) but corresponds to a gravitational effective action, the definition (2.9) shows that

$$\Psi(g) = -\ln Z_{\text{mat+gh}}(g) \quad (2.30)$$

and thus

$$t^{ab}(g) = \langle T^{ab} \rangle_{\text{mat+gh}} \quad (2.31)$$

is the vacuum expectation value of the stress-energy tensor T^{ab} of the matter plus ghost QFT in the fixed background geometry g . The gravitational effective action is thus always determined by the expectation value of the trace of the stress-energy tensor.

For example, if the matter theory is a CFT, the trace of the stress-energy tensor is zero classically and its possible non-zero value at the quantum level is entirely due to a quantum anomaly. An anomaly must always be a local functional and the only possibility for the trace anomaly in two dimensions consistent with dimensional analysis and diffeomorphism invariance is $t = a + bR$ for some constants a and b . The constant a can always be absorbed in a redefinition of the cosmological constant and thus can be discarded. From (3.3), the gravitational effective action must be the Liouville action, as indicated in (2.18). The precise coefficient $b = (26 - c)/(24\pi)$ can be computed in various ways, for example in the flat space limit. We shall generalize this reasoning for the non-conformal model (1.7) in Section 4.2.

Finally, let us note that the gravitational dressing factors satisfy exponentiated versions of the cocycle conditions that follow immediately from their definition (2.15),

$$D_{\sigma_1 \dots \sigma_n}(g_1, g_2) = \frac{1}{D_{\sigma_1 \dots \sigma_n}(g_2, g_1)}, \quad (2.32)$$

$$D_{\sigma_1 \dots \sigma_n}(g_1, g_3) = D_{\sigma_1 \dots \sigma_n}(g_1, g_2) D_{\sigma_1 \dots \sigma_n}(g_2, g_3). \quad (2.33)$$

3 Functionals and variations

We are now going to derive a set identities for the variation under change of metric of various functionals. All these identities will then be applied in Section 4 to compute the gravitational effective action and gravitational dressing of the model (1.7). The technical derivations that we present below and in the Appendices are not necessary in understanding Section 4 and thus may be skipped on a first reading.

3.1 Building blocks for gravitational actions

3.1.1 The area functional

The simplest functional that can enter in a gravitational effective action is the area, or more generally any function of the area,

$$S_f(g_0, g) = f(A) - f(A_0). \quad (3.1)$$

This is associated with a term

$$t_f = 4\pi f'(A) \quad (3.2)$$

in the trace of the stress-energy tensor.

3.1.2 The Liouville action

The variation of the Liouville action (1.2) with respect to σ ,

$$\delta S_L(g_0, g) = \int_{\Sigma_h} d^2x \sqrt{g} R \delta \sigma. \quad (3.3)$$

yields the term

$$t_L = 2\pi R \quad (3.4)$$

in the trace of the stress-energy tensor.

3.1.3 The Mabuchi action

The Mabuchi action, defined in (1.4), can also be written as

$$S_M(g_0, g) = \int_{\Sigma_h} \left[-4i\pi(1-h)\partial\phi \wedge \bar{\partial}\phi + \left(\frac{8\pi(1-h)}{A_0} - R_0 \right) \phi \omega_0 + \frac{2}{A} \omega \ln \frac{\omega}{\omega_0} \right] \quad (3.5)$$

$$= \int_{\Sigma_h} \left[4\pi(1-h)\phi \left(\frac{\omega_0}{A_0} + \frac{\omega}{A} \right) - \phi R_0 \omega_0 + \frac{2}{A} \omega \ln \frac{\omega}{\omega_0} \right]. \quad (3.6)$$

Let us note that this definition involves metrics g_0 and g that do not necessarily have the same area. In the mathematical literature, the Mabuchi action has been defined and used only for metrics having the same area or more generally, in higher dimensions, for metrics in the same Kähler class. The generalized definition (3.5) is natural in our case, because it is in this way that it will appear in the calculations of the gravitational effective actions in Section 4. In particular, it satisfies the cocycle identities (1.5) and (1.6) for any metrics g_0 and g , including when g_0 and g do not have the same areas. It would be interesting to know if the higher dimensional versions of the action can also be suitably generalized as functions of the Kähler moduli consistently with the cocycle identities.

The variation of the Mabuchi action with respect to g yields

$$\delta S_M = 2 \frac{\delta A}{A} - \int_{\Sigma_h} d^2x \sqrt{g} (R - \bar{R}) \delta \phi, \quad (3.7)$$

where the constant \bar{R} is the average Ricci scalar, or equivalently the Ricci scalar for the metric of constant scalar curvature and area A ,

$$\bar{R} = \frac{1}{A} \int_{\Sigma_h} d^2x \sqrt{g} R = \frac{8\pi(1-h)}{A}. \quad (3.8)$$

Comparing with (2.28), we see that the trace of the stress-energy for the Mabuchi action satisfies

$$\Delta t_M = \frac{8\pi}{A} (R - \bar{R}), \quad \int_{\Sigma_h} d^2x \sqrt{g} t_M = 8\pi. \quad (3.9)$$

The *Ricci potential* ψ is usually defined by the conditions

$$\Delta \psi = R - \bar{R}, \quad (3.10)$$

$$\int_{\Sigma_h} d^2x \sqrt{g} \psi = 0, \quad (3.11)$$

which always have a unique solution. Comparing (3.10), (3.11) and (3.9), we find

$$t_M = \frac{8\pi}{A} (\psi + 1). \quad (3.12)$$

3.1.4 The Aubin-Yau action

The Aubin-Yau action is defined by

$$S_{AY}(g_0, \phi) = - \int_{\Sigma_h} d^2x \sqrt{g_0} \left(\frac{1}{4} \phi \Delta_0 \phi - \frac{\phi}{A_0} \right) \quad (3.13)$$

$$= -\frac{i}{2} \int_{\Sigma_h} \partial \phi \wedge \bar{\partial} \phi + \frac{1}{A_0} \int_{\Sigma_h} \phi \omega_0 \quad (3.14)$$

$$= \frac{1}{2} \int_{\Sigma_h} \phi \left(\frac{\omega_0}{A_0} + \frac{\omega}{A} \right). \quad (3.15)$$

As in the case of the Mabuchi action, we give here a definition valid for metrics having different areas, slightly generalizing the functional used in the mathematical literature.

It is important to realize that the Aubin-Yau action is not a functional of the metric g , because it is not invariant under constant shifts of ϕ ,

$$S_{\text{AY}}(g_0, \phi + \text{constant}) = S_{\text{AY}}(g_0, \phi) + \text{constant} . \quad (3.16)$$

This is why we have been careful in (3.13) to write $S_{\text{AY}}(g_0, \phi)$ and not $S_{\text{AY}}(g_0, g)$. In particular, the variation of S_{AY} with respect to ϕ ,

$$\delta S_{\text{AY}} = \frac{1}{A} \int_{\Sigma_h} d^2x \sqrt{g} \delta \phi , \quad (3.17)$$

is not of the form (2.28).

However, S_{AY} does satisfy cocycle identities, in the form

$$S_{\text{AY}}(g_1, \phi_{1,3}) = S_{\text{AY}}(g_1, \phi_{1,2}) + S_{\text{AY}}(g_2, \phi_{2,3}) , \quad (3.18)$$

with $\omega_2 = (A_2/A_1)\omega_1 + iA_2\partial\bar{\partial}\phi_{1,2}$, $\omega_3 = (A_3/A_2)\omega_2 + iA_3\partial\bar{\partial}\phi_{2,3}$ and $\phi_{1,3} = \phi_{1,2} + \phi_{2,3}$. Moreover, it is very easy to add simple terms to S_{AY} to make it a well-defined functional of the metric. For example, a combination

$$S_{\text{AY}}(g_0, \phi) - \frac{1}{\hat{A}} \int_{\Sigma_h} \sqrt{\hat{g}} \phi , \quad (3.19)$$

for any fixed metric \hat{g} of area \hat{A} , is invariant under constant shifts of ϕ and satisfies the cocycle identities. The metric \hat{g} could be for example the metric of constant curvature, or any other canonically defined metric on Σ_h . As we shall see in the following, it is in the form (3.19) that the Aubin-Yau action contributes to the gravitational effective action of the model (1.7). Similarly, factors of the form

$$e^{K(\phi(x) - S_{\text{AY}}(g_0, \phi))} , \quad (3.20)$$

for some constant K , are invariant under constant shifts of ϕ and satisfy the consistency conditions (2.32) and (2.33). We shall demonstrate in Section 4 that these factors do contribute to the gravitational dressing of the operators (1.8), see eq. (4.32).

3.2 Variation formulas

We are now going to present functionals Ψ satisfying (2.29) for the various actions introduced in the previous subsection, as well as some natural generalizations.

We denote by ψ_i , $i \geq 0$, the (real) eigenfunctions of the laplacian with eigenvalues λ_i , $0 = \lambda_0 < \lambda_1$, $\lambda_i \leq \lambda_{i+1}$. They are normalized such that

$$\langle \psi_i | \psi_j \rangle = \int_{\Sigma_h} d^2x \sqrt{g} \psi_i \psi_j = \delta_{ij}. \quad (3.21)$$

In particular, the zero mode is

$$\psi_0 = 1/\sqrt{A}. \quad (3.22)$$

The ζ -function is defined by

$$\zeta(s) = \sum_{i>0} \frac{1}{\lambda_i^s}. \quad (3.23)$$

The series in (3.23) converges absolutely for $\text{Re } s > 1$. By analytic continuation, it defines an analytic function ζ for all complex values of s , except at $s = 1$ where it has a simple pole with residue $A/(4\pi)$.

3.2.1 Partition function, determinant and the Liouville action

As is well-known, the Liouville action can be written as

$$S_L(g_0, g) = \Psi_L(g) - \Psi_L(g_0) \quad (3.24)$$

where the functional Ψ_L is given by

$$\Psi_L(g) = \frac{24\pi}{c} \ln Z_{\text{CFT}}(g) \quad (3.25)$$

in terms of the partition function Z_{CFT} of any conformal field theory of central charge c in the background metric g . Equation (2.18) is a special case of (3.25) with central charge $c = 26$ taking into account the contribution from the ghosts.

The simple case that will be useful for us corresponds to the $c = 1$ CFT of a free massless scalar field, which yields

$$\Psi_L(g) = -12\pi \ln \frac{\det' \Delta_g}{A(g)}, \quad (3.26)$$

where $A(g)$ is the area and

$$\det' \Delta_g = e^{-\zeta'(0)} \quad (3.27)$$

is the infinite dimensional determinant of the laplacian with the zero mode excluded, defined in terms of the analytic continuation of the ζ function (3.23). Let us emphasize that the factor $1/A$ in (3.26) is crucial for (3.24) to be valid.

The above results are standard and we shall not repeat the well-known proofs here.

3.2.2 Green's functions and the Aubin-Yau action

Let $G(x, y; g)$ be the Green's function for the laplacian Δ in the metric g . It is uniquely defined by the conditions

$$\Delta_x G(x, y; g) = \frac{\delta(x - y)}{\sqrt{g}} - \frac{1}{A}, \quad (3.28)$$

$$\int_{\Sigma_h} d^2x \sqrt{g} G(x, y; g) = 0 \quad (3.29)$$

and can be expressed in terms of the eigenfunctions and eigenvalues of the laplacian as

$$G(x, y; g) = \sum_{i>0} \frac{\psi_i(x)\psi_i(y)}{\lambda_i}. \quad (3.30)$$

We are going to prove the following identity relating the variation of the Green's function to the Aubin-Yau action,

$$G(x, y; g) - G(x, y; g_0) = \frac{1}{2} \left(\phi(x) + \phi(y) \right) - S_{\text{AY}}(g_0, \phi). \quad (3.31)$$

Let us note that the left-hand side of the above equation is a well-defined functional of the metric g and thus the right-hand side must be invariant under constant shifts in ϕ , which is indeed the case from the transformation rule (3.16).

There are many ways to prove (3.31). The simplest is to check that the conditions (3.28) and (3.29) are satisfied for $G(x, y; g)$ given by (3.31), if they are satisfied for $G(x, y; g_0)$. To check (3.28) we use (1.17) and (1.3),

$$\Delta_x G(x, y; g) = e^{-2\sigma} \left(\Delta_{0x} G(x, y; g_0) + \frac{1}{2} \Delta_0 \phi(x) \right) \quad (3.32)$$

$$= e^{-2\sigma} \left(\frac{\delta(x - y)}{\sqrt{g_0}} - \frac{1}{A_0} + \frac{1}{A_0} - \frac{e^{2\sigma}}{A} \right) \quad (3.33)$$

$$= \frac{\delta(x - y)}{\sqrt{g}} - \frac{1}{A}. \quad (3.34)$$

To check (3.29), we use $\sqrt{g} = e^{2\sigma} \sqrt{g_0}$ and (1.3) to write

$$\begin{aligned} \int_{\Sigma_h} d^2x \sqrt{g} G(x, y; g) &= \int_{\Sigma_h} d^2x \sqrt{g_0} \left(\frac{A}{A_0} - \frac{1}{2} A \Delta_0 \phi(x) \right) \left(G(x, y; g_0) + \frac{1}{2} \phi(x) \right) \\ &\quad + \frac{A}{2} \phi(y) - A S_{\text{AY}}(g_0, \phi). \end{aligned} \quad (3.35)$$

We then integrate by part the term containing $\Delta_0 \phi$, use the conditions (3.28) and (3.29) for $g = g_0$ and the definition of the Aubin-Yau action (3.13) to get (3.29) for g .

An alternative derivation would be to prove the infinitesimal version of (3.31),

$$\delta G(x, y; g) = \frac{1}{2} \left(\delta \phi(x) + \delta \phi(y) \right) - \int_{\Sigma_h} d^2x \sqrt{g} \delta \phi. \quad (3.36)$$

This can be done starting from the defining equation (3.30) and using the quantum mechanical perturbation theory formulas for the infinitesimal variations of the eigenvalues and eigenfunctions of the laplacian

$$\delta \lambda_i = -2 \lambda_i \langle \psi_i | \delta \sigma | \psi_i \rangle, \quad (3.37)$$

$$\delta \psi_i = -\langle \psi_i | \delta \sigma | \psi_i \rangle \psi_i - 2 \sum_{j \neq i} \frac{\lambda_i}{\lambda_i - \lambda_j} \langle \psi_j | \delta \sigma | \psi_i \rangle \psi_j. \quad (3.38)$$

Deriving (3.36) is then a straightforward calculation that we let to the reader. Let us note that the formulas (3.37) and (3.38) are valid only when the spectrum is non-degenerate, but this is of course almost always true and (3.31) follows in all cases by continuity.

When $x \rightarrow y$, the Green's function has the usual short distance logarithmic divergence. This divergence can be subtracted in a diffeomorphism invariant way, by using the geodesic distance function d_g and introducing an arbitrary, metric-independent, length scale ℓ . The renormalized Green's function at coincident points $G_R(x)$ is then defined by

$$G_R(x; g) = \lim_{y \rightarrow x} \left[G(x, y; g) + \frac{1}{2\pi} \ln \frac{d_g(x, y)}{\ell} \right]. \quad (3.39)$$

We shall also use later an essentially equivalent definition in terms of ζ functions, see (B.5) and (B.6). This formula together with (3.31) immediately yield

$$G_R(x; g) - G_R(x, g_0) = \phi(x) - S_{\text{AY}}(g_0, \phi) + \frac{\sigma(x)}{2\pi}, \quad (3.40)$$

the last term $\sigma/(2\pi)$ coming from the variation of the geodesic distance under rescaling of the metric.

The formulas (3.31) and (3.40) could also be derived starting from standard general formulas for the Weyl rescaling of the Green's function that can be found for example in [9].

3.2.3 The Ricci potential and the Mabuchi and Aubin-Yau actions

The Ricci potential associated with a given metric was defined by the conditions (3.10) and (3.11). The difference between Ricci potentials associated with different metrics is given by an interesting formula involving both the Aubin-Yau and Mabuchi actions,

$$\psi(x; g) - \psi(x; g_0) = 2\sigma(x) + 4\pi(1 - h)(\phi(x) - S_{\text{AY}}(g_0, \phi)) - \frac{1}{2} S_{\text{M}}(g_0, g). \quad (3.41)$$

To prove this identity, we first check that

$$\Delta(\psi(x; g_0) + 2\sigma(x) + 4\pi(1-h)\phi(x)) = R - \bar{R} \quad (3.42)$$

using $\Delta_0\psi(x; g_0) = R_0 - \bar{R}_0$, (1.18) and (1.3). This implies that

$$\psi(x; g) - \psi(x; g_0) = 2\sigma(x) + 4\pi(1-h)\phi(x) + C(g_0, g), \quad (3.43)$$

for some x -independent functional $C(g_0, g)$. This functional is then obtained by computing the integral of (3.43) and imposing the condition (3.11) for both $\psi(x; g)$ and $\psi(x; g_0)$, using in particular (1.3).

Let us note that if \bar{g} is the metric of constant scalar curvature, $\psi(x; \bar{g}) = 0$. Equation (3.41) then yields the Ricci potential for any metric $g = e^{2\sigma}\bar{g}$,

$$\psi(x; g) = 2\sigma(x) + 4\pi(1-h)(\phi(x) - S_{\text{AY}}(\bar{g}, \phi)) - \frac{1}{2}S_{\text{M}}(\bar{g}, g). \quad (3.44)$$

Let us also note that the Ricci potential is given by the following explicit integral formula

$$\psi(x; g) = \int_{\Sigma_h} dy \sqrt{g(y)} G(x, y; g) R(y) \quad (3.45)$$

in terms of the Green's function. Equation (3.41) could also be straightforwardly derived from this integral representation by using (3.31).

3.2.4 The Polyakov functional and the Mabuchi action

The next functional we wish to study is Polyakov's effective action

$$\Psi_{\text{P}}(g) = \frac{1}{4} \int_{\Sigma_h \times \Sigma_h} d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} R(x) G(x, y; g) R(y). \quad (3.46)$$

This is the famous non-local action $\iint R \Delta^{-1} R$ introduced by Polyakov in [1] to describe the partition function of a CFT. However, when the theory is formulated at finite area, it turns out that the variation of Ψ_{P} crucially involves the Mabuchi action on top of the Liouville action,

$$\Psi_{\text{P}}(g) - \Psi_{\text{P}}(g_0) = S_{\text{L}}(g_0, g) - 2\pi(1-h)S_{\text{M}}(g_0, g). \quad (3.47)$$

In particular, Ψ_{P} as defined in (3.46) is *not* suitable to describe the partition function of a CFT, whose variation (3.24) always yields the Liouville action without any additional term. When $A \rightarrow \infty$, the Mabuchi term vanishes because in this limit (1.3) shows that ϕ scales like $1/A$.

The derivation of (3.47) is a bit tedious but completely straightforward. One starts by expressing $\Psi_{\text{P}}(g)$ in terms of quantities related to g_0 , by using equations

(1.18), (3.31), (3.28), (3.29) and (1.3). Terms involving $\Delta_0\sigma$ or $\Delta_0\phi$ are dealt with by integrating by part and using repeatedly (3.28). The genus h enters from integrals of the form (1.16).

A very simple expression for the functional (3.46) can also be obtained in terms of the Ricci potential ψ defined in (3.10) and (3.11), by replacing R by $\bar{R} + \Delta\psi$ in (3.46),

$$\Psi_P(g) = \frac{1}{4} \int_{\Sigma_h} d^2x \sqrt{g} \psi \Delta\psi. \quad (3.48)$$

This formula, together with (3.41), is another good starting point to derive (3.47).

3.2.5 The integrated Green's function

The last non-trivial functional that we shall need in Section 4 is the integral of the Green's function at coincident points,

$$\Psi_G(g) = \frac{1}{A(g)} \int_{\Sigma_h} d^2x \sqrt{g} G_R(x; g). \quad (3.49)$$

By using (1.3) and (3.40), as well as the explicit formula (3.13) for S_{AY} , we easily obtain

$$\Psi_G(g) - \Psi_G(g_0) = \frac{1}{8\pi} \int_{\Sigma_h} d^2x \sqrt{g_0} \left[-2\pi\phi\Delta_0\phi + \frac{4}{A}\sigma e^{2\sigma} - 4\pi\phi\Delta_0 G_R(x; g_0) \right]. \quad (3.50)$$

This formula has also been derived and used in [10] using different methods.

For our purposes, we need to go further and evaluate ΔG_R . We present first the cases of the sphere and the torus, for which a very elementary calculation is available.

The case of the sphere

On the sphere endowed with the round metric \bar{g} , the $SO(3)$ invariance implies that the scalar function $G_R(x; \bar{g})$ must be a constant. Applying the Laplace operator on the equation (3.40) with $g_0 = \bar{g}$ thus yields

$$\Delta G_R(x; g) = \Delta\phi + \frac{1}{2\pi}\Delta\sigma = \frac{1}{4\pi}\Delta\psi, \quad (3.51)$$

where the Ricci potential on the sphere is given by (3.44) for $h = 0$. Using (3.10) we thus conclude that

$$\Delta G_R = \frac{R - \bar{R}}{4\pi} = \frac{R}{4\pi} - \frac{2}{A} \quad \text{on the sphere.} \quad (3.52)$$

Plugging this result for $g = g_0$ into (3.50) and comparing the result with the definition (1.4) of the $h = 0$ Mabuchi action then yields

$$\Psi_G(g) - \Psi_G(g_0) = \frac{1}{8\pi} S_M(g_0, g). \quad (3.53)$$

The case of the torus

On the torus and for $g_0 = \bar{g}$ the flat metric, $G_R(x; \bar{g})$ is a constant by translation invariance. The first equation in (3.51) is thus still valid, but the formula (3.44) for the Ricci potential in the case $h = 1$ yields in the present case

$$\Delta G_R(x; g) = \frac{1}{4\pi} \Delta \psi + \Delta \phi = \frac{R}{4\pi} - \frac{2}{A} + \frac{2}{\bar{A}} e^{-2\sigma}, \quad (3.54)$$

where \bar{A} is the area for the flat metric \bar{g} and we have used (1.3) and (1.17). Let us introduce the usual coordinates (z, \bar{z}) on the torus, with the identifications $z \equiv z + 1$ and $z \equiv z + \tau$ for a complex structure modulus τ such that $\text{Im } \tau > 0$. The flat metric of area \bar{A} is simply

$$\bar{g} = \frac{\bar{A}}{\text{Im } \tau} |dz|^2 \quad (3.55)$$

and thus (3.54) can be rewritten in the nice form

$$\Delta G_R(x; g) = \frac{R}{4\pi} - \frac{2}{A} + \frac{2}{\sqrt{g}} \frac{1}{\text{Im } \tau} = \frac{R}{4\pi} - \frac{2}{A} + 2 \frac{\sqrt{g^c}}{\sqrt{g}}, \quad (3.56)$$

where

$$g^c = \frac{1}{\text{Im } \tau} |dz|^2 \quad (3.57)$$

is the so-called *canonical* metric on the torus, the flat metric normalized to have unit area.

Plugging (3.56) into (3.50) and comparing with the definitions of the $h = 1$ Mabuchi (1.4) and Aubin-Yau (3.13) actions, we obtain

$$\Psi_G(g) - \Psi_G(g_0) = \frac{1}{8\pi} S_M(g_0, g) + S_{AY}(g_0, \phi) - \int_{\Sigma_h} d^2x \sqrt{g^c} \phi. \quad (3.58)$$

The Aubin-Yau term in the above formula is of the form (3.19).

The case of a general compact Riemann surface

For an arbitrary genus h , (3.53) and (3.58) generalize to

$$\Psi_G(g) - \Psi_G(g_0) = \frac{1}{8\pi} S_M(g_0, g) + h \left(S_{AY}(g_0, \phi) - \int_{\Sigma_h} d^2x \sqrt{g^c} \phi \right). \quad (3.59)$$

The *canonical* metric g^c on Σ_h is defined as follows. We introduce a canonical basis of homology cycles (α_i, β_i) , $1 \leq i \leq h$, on Σ_h , with non-trivial intersection numbers $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. We also introduce a basis of holomorphic one-forms λ_i , $1 \leq i \leq h$, such that

$$\oint_{\alpha_i} \lambda_j = \delta_{ij}. \quad (3.60)$$

The Kähler form of the canonical metric is then defined to be

$$\omega^c = \frac{i}{2h} (\text{Im } \tau)_{ij}^{-1} \lambda_i \wedge \bar{\lambda}_j \quad (3.61)$$

in terms of the period matrix

$$\tau_{ij} = \oint_{\beta_i} \lambda_j. \quad (3.62)$$

The formula (3.61) indeed defines a metric because the matrix τ is symmetric and $\text{Im } \tau$ is positive-definite; in a coordinate system (z, \bar{z}) in which

$$\lambda_i = l_i(z) dz, \quad (3.63)$$

we have

$$2g_{z\bar{z}}^c = \sqrt{g^c} = \frac{1}{h} (\text{Im } \tau)_{ij}^{-1} l_i(z) \bar{l}_j(\bar{z}) > 0. \quad (3.64)$$

Let us note that the definition (3.61) is of course independent of the choice of basis of holomorphic one-forms satisfying (3.60), and that the overall normalization is chosen in such a way that g^c has unit area, as can be easily shown using Riemann bilinear relations. The canonical metric can also be interpreted as being the pull-back of the trivial metric on the Jacobian variety of Σ_h under the Abel-Jacobi map.

Using (3.50), equation (3.59) follows from the following general formula for the laplacian of G_R ,

$$\Delta G_R(x; g) = \frac{R}{4\pi} - \frac{2}{A} + 2h \frac{\sqrt{g^c}}{\sqrt{g}}. \quad (3.65)$$

If we try to repeat the reasoning made at genus zero and one to prove this formula, applying the Laplace operator on (3.40) and using (1.3) and (1.17), we get a nice transformation rule

$$\Delta G_R(x; g) - \frac{R}{4\pi} + \frac{2}{A} = e^{-2\sigma} \left(\Delta G_R(x; g_0) - \frac{R_0}{4\pi} + \frac{2}{A_0} \right). \quad (3.66)$$

However, this result would yield ΔG_R for any metric if it were known for a particular metric on Σ_h , which is not the case for $h \geq 2$. We thus have to proceed in a different way.

Let us consider the function

$$\mathbf{G}_R(x, y; g) = G(x, y; g) + \frac{1}{2\pi} \ln \frac{d_g(x, y)}{\ell}. \quad (3.67)$$

It is symmetric, $\mathbf{G}_R(x, y; g) = \mathbf{G}_R(y, x; g)$, and regular at $x = y$. The definition (3.39) is equivalent to

$$G_R(x; g) = \mathbf{G}_R(x, x; g). \quad (3.68)$$

Let us pick a coordinate system (x^a) for which, in some open set, the metric has the simple form

$$g_{ab} = e^{2\sigma} \delta_{ab} \quad (3.69)$$

and thus the laplacian reads

$$\Delta = -e^{2\sigma} \partial_a \partial_a. \quad (3.70)$$

Taking the partial derivatives of (3.68) and using the symmetry of \mathbf{G}_R we get

$$\partial_a \partial_b G_R(x; g) = 2(\partial_{1a} \partial_{1b} + \partial_{1a} \partial_{2b}) \mathbf{G}_R(x, x; g), \quad (3.71)$$

where ∂_{1a} and ∂_{2a} are partial derivatives with respect to the first and second argument in $\mathbf{G}_R(x, y; g)$. This implies in particular that

$$\begin{aligned} \Delta G_R(x; g) = 2 \lim_{y \rightarrow x} & \left[\Delta_1 G(x, y; g) + \frac{1}{2\pi} \Delta_1 \ln d_g(x, y) \right. \\ & \left. - e^{-2\sigma} \partial_{1a} \partial_{2a} G(x, y; g) - \frac{1}{2\pi} e^{-2\sigma} \partial_{1a} \partial_{2a} \ln d_g(x, y) \right], \end{aligned} \quad (3.72)$$

where Δ_1 acts on the first argument x . The terms involving the geodesic distance can be evaluated using the standard short distance expansion in the coordinate system (3.69),

$$\begin{aligned} 2 \ln d_g(x, y) = & \ln((y^a - x^a)(y^a - x^a)) + 2\sigma(x) + (y^a - x^a) \partial_a \sigma(x) \\ & - \frac{1}{12} (y^a - x^a)(y^a - x^a) \partial_b \sigma(x) \partial_b \sigma(x) + \frac{1}{6} (y^a - x^a)(y^b - x^b) \partial_a \sigma(x) \partial_b \sigma(x) \\ & + \frac{1}{3} (y^a - x^a)(y^b - x^b) \partial_a \partial_b \sigma(x) + \mathcal{O}((y - x)^3), \end{aligned} \quad (3.73)$$

which yields

$$\frac{1}{2\pi} \Delta_1 \ln d_g(x, y) \underset{x \rightarrow y}{=} -\frac{\delta(x - y)}{\sqrt{g}} + \frac{R}{12\pi}, \quad (3.74)$$

$$\frac{1}{2\pi} e^{-2\sigma} \partial_{1a} \partial_{2a} \ln d_g(x, y) \underset{x \rightarrow y}{=} -\frac{\delta(x - y)}{\sqrt{g}} - \frac{R}{24\pi}. \quad (3.75)$$

Using (3.28), we thus obtain

$$\Delta G_R(x; g) = 2 \lim_{x \rightarrow y} \left[\frac{\delta(x - y)}{\sqrt{g}} + \frac{R}{8\pi} - \frac{1}{A} - e^{-2\sigma} \partial_{1a} \partial_{2a} G(x, y; g) \right]. \quad (3.76)$$

Equation (3.65) then follows from the identity

$$\partial_{1a} \partial_{2a} G(x, y; g) \underset{x \rightarrow y}{=} \delta(x - y) - (\text{Im } \tau)_{ij}^{-1} l_i \bar{l}_j. \quad (3.77)$$

This identity is a direct consequence of a standard formula (A.2) for the mixed partial derivatives of the Green's function in a coordinate system (3.69). This formula can be found for example in [9]. Since we have not been able to find a simple derivation in the literature, we have included one in Appendix A.

4 Application to the massive scalar field

We are now going to apply the results of the previous Section to study the model (1.7). In order to compute both the effective gravitational action and the gravitational dressing of the operators (1.8) at the same time, we consider the generalized partition function

$$Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g) = \int \mathcal{D}X e^{\sum_{i=1}^n k_i X(x_i) - S_{\text{mat}}(X; q, m; g)}, \quad (4.1)$$

where \mathbf{k} and \mathbf{x} denote collectively all the k_i s and x_i s. This generalized partition function contains all the information we need. Indeed, the usual matter partition function is simply

$$Z_{\text{mat}}(q, m; g) = Z_{\text{mat}}(q, m; \mathbf{k} = 0, \mathbf{x}; g). \quad (4.2)$$

Taking into account the contribution from the ghost CFT, the gravitational effective action (2.9) is given by

$$S_{\text{grav}}(g_0, g; q, m) = \frac{13}{12\pi} S_L(g_0, g) - \ln \frac{Z_{\text{mat}}(q, m; g)}{Z_{\text{mat}}(q, m; g_0)}. \quad (4.3)$$

The correlators for the operators (1.8), defined in (2.13), are given by

$$\langle \mathcal{O}_{k_1}(x_1) \cdots \mathcal{O}_{k_n}(x_n) \rangle_{\text{mat}, g} = \frac{Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g)}{Z_{\text{mat}}(q, m; g)}. \quad (4.4)$$

Finally, the gravitational dressing (2.15) is

$$D_{\mathcal{O}_{k_1} \cdots \mathcal{O}_{k_n}}(\mathbf{x}, \mathbf{k}; q, m; g_0, g) = \frac{Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g)}{Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g_0)} e^{2 \sum_{i=1}^n \sigma(x_i) - \frac{13}{12\pi} S_L(g_0, g) + S_{\text{grav}}(g_0, g; q, m)}. \quad (4.5)$$

We are going to study the model *at leading non-trivial order when the mass parameter is small*. If we work at fixed area, as in (2.3), this means that the condition

$$m^2 A \ll 1 \quad (4.6)$$

must be satisfied. If we wish to integrate over areas, then we choose the cosmological constant to be much larger than m^2 ,

$$m^2 \ll \mu. \quad (4.7)$$

It is important to understand that this mass expansion is non-perturbative. Indeed, the mass term in (1.7) is not a well-defined operator in the $m = 0$ CFT.

4.1 Direct calculation

4.1.1 The partition function

We introduce the Green's function $G(x, y; m; g)$ for the massive scalar field, characterized by the condition

$$(\Delta_x + m^2)G(x, y; m; g) = \frac{\delta(x - y)}{\sqrt{g}} \quad (4.8)$$

and expressed explicitly in terms of the eigenfunctions and eigenvalues of the laplacian as

$$G(x, y; m; g) = \sum_{i \geq 0} \frac{\psi_i(x)\psi_i(y)}{\lambda_i + m^2}. \quad (4.9)$$

The path integral (4.1) is gaussian, so a direct calculation of the partition function in terms of the above Green's function is completely straightforward. The only subtlety is that we have to subtract the infinities due to the self-contractions. We proceed as usual ("normal ordering"), in a diffeomorphism invariant way, by replacing each instance of $G(x, x; m; g)$ by its renormalized version $G_R(x; m; g)$, defined along the lines of (3.39),

$$G_R(x; m; g) = \lim_{y \rightarrow x} \left[G(x, y; m; g) + \frac{1}{2\pi} \ln \frac{d_g(x, y)}{\ell} \right]. \quad (4.10)$$

The parameter ℓ plays the rôle of an arbitrary renormalization scale. We also define the renormalized determinant of the euclidean Klein-Gordon operator with the ζ -function prescription, along the lines of (3.23) and (3.27),

$$\det(\Delta + m^2) = e^{-\zeta'_m(0)}, \quad (4.11)$$

with

$$\zeta_m(s) = \sum_{i \geq 0} \frac{1}{(\lambda_i + m^2)^s}. \quad (4.12)$$

The partition function (4.1) then reads

$$\begin{aligned} Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g) = & \frac{1}{\sqrt{\det(\Delta + m^2)}} \\ & \exp \left[2\pi \sum_{i \neq j} k_i k_j G(x_i, x_j; m; g) + 2\pi \sum_i k_i^2 G_R(x_i; m; g) - \right. \\ & \quad \frac{q}{2} \sum_i k_i \int_{\Sigma_h} d^2x \sqrt{g} G(x, x_i; m; g) R(x) + \\ & \quad \left. \frac{q^2}{32\pi} \int_{\Sigma_h \times \Sigma_h} d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} R(x) G(x, y; m; g) R(y) \right]. \quad (4.13) \end{aligned}$$

4.1.2 The small mass expansion

To pick up the leading non-trivial contributions at small m , we need to approximate the Green's functions (4.9), (4.10) and the determinant (4.11). The small mass expansion of the Green's functions is straightforward to obtain. Taking care of the zero mode (3.22) and using (3.30) and (3.39) we get

$$G(x, y; m; g) = \frac{1}{m^2 A} + G(x, y; g) + m^2 G_2(x, y; g) + \mathcal{O}(m^4), \quad (4.14)$$

$$G_R(x; m; g) = \frac{1}{m^2 A} + G_R(x; g) + m^2 G_2(x, x; g) + \mathcal{O}(m^4), \quad (4.15)$$

where $G_2(x, y; g)$ is defined by

$$G_2(x, y; g) = - \sum_{i>0} \frac{\psi_i(x) \psi_i(y)}{\lambda_i^2}. \quad (4.16)$$

On the other hand, from (4.11) and the small mass expansion of the ζ -function (4.12), we get

$$\ln \det(\Delta + m^2) = \ln(m^2 \det' \Delta) + m^2 \lim_{s \rightarrow 0} [\zeta(s+1) + s \zeta'(s+1)] + \mathcal{O}(m^4). \quad (4.17)$$

The second term on the right hand side gives a renormalized version of the formal sum $\sum_{i>0} 1/\lambda_i$ or equivalently of the integrated Green's function $\int d^2x \sqrt{g} G(x, x; g)$. Another renormalized version of the same functional is given by $A \Psi_G(g)$ defined in (3.49). The difference between these two definitions of $\sum_{i>0} 1/\lambda_i$, based on two different renormalization schemes (explicit subtraction of the UV divergence in the definition (3.49) using G_R (3.39) or ζ -function in (4.17)) must be a finite local counterterm, which presently can only correspond to a finite shift of the cosmological constant. Indeed, we show explicitly in Appendix B that

$$\lim_{s \rightarrow 0} [\zeta(s+1) + s \zeta'(s+1)] = A \left(\Psi_G(g) + \frac{1}{2\pi} (\ln \ell - \ln 2 + \gamma) \right), \quad (4.18)$$

where $\gamma \simeq 0.577$ is Euler's constant.

4.1.3 The gravitational action

Plugging the expansions (4.14) and (4.17) into (4.13) for $\mathbf{k} = 0$ we obtain

$$\begin{aligned} -\ln Z_{\text{mat}}(q, m; g) &= -\frac{2\pi(1-h)^2 q^2}{m^2 A} + \frac{1}{2} \ln(m^2 \det' \Delta) - \frac{q^2}{32\pi} \Psi_P(g) \\ &\quad + \frac{1}{2} A m^2 \left(\Psi_G(g) + \frac{1}{2\pi} (\ln \ell - \ln 2 + \gamma) \right) \\ &\quad - \frac{q^2 m^2}{32\pi} \int_{\Sigma_h \times \Sigma_h} d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} R(x) G_2(x, y; g) R(y) + \mathcal{O}(m^4). \end{aligned} \quad (4.19)$$

Case $q \neq 0$ The *leading* non-trivial contributions are then given by the first line in (4.19). From (3.24), (3.26) and (3.47), we obtain the leading gravitational action

$$S_{\text{grav}}(g_0, g; q, m) = -\frac{2\pi(1-h)^2 q^2}{m^2} \left(\frac{1}{A} - \frac{1}{A_0} \right) + \frac{1}{2} \ln \frac{A}{A_0} \\ + \frac{25-3q^2}{24\pi} S_{\text{L}}(g_0, g) + \frac{(1-h)q^2}{4} S_{\text{M}}(g_0, g). \quad (4.20)$$

The terms in $1/A$ in the action come from the integration over the zero mode of the scalar field X . These terms are seen in R^2 gravity models [11], because an R^2 term in the bare gravitational action is equivalent to the model (1.7) with no kinetic term and an imaginary q . For real q , the effect of these terms is to enhance the contribution of surfaces of small areas, consistently with our approximation (4.6). In the $m \rightarrow 0$ CFT limit, the integral over the zero mode imposes the “neutrality” condition $q = 0$.

The coefficient $3q^2 - 25 = 1 + 3q^2 - 26$ in front of the Liouville action is the total central charge of the conformal $m = 0$ model coupled with the ghost CFT. This coefficient can be made to vanish, either by adjusting q , or by coupling the system to a spectator CFT of central charge c such that

$$1 + 3q^2 + c - 26 = 0. \quad (4.21)$$

For $h \neq 1$, the non-trivial part of the gravitational action is then entirely given by the Mabuchi action. If q is real and $h = 0$, the Mabuchi action comes with the right sign to define non-perturbatively a new two-dimensional quantum gravity model, because S_{M} is bounded from below and convex on the space of metrics. When $h \geq 2$, q must be imaginary. In this case positivity is lost and we must work at fixed area for the small mass approximation to make sense, because of the $1/A$ term in the action. If $h \geq 2$ and q is real, higher order terms in the small mass expansion cannot be neglected non-perturbatively, since otherwise the gravitational action wouldn’t be bounded from below.

Case $q = 0$ The leading non-trivial contribution to the gravitational action is then given by the second line in (4.19). From (3.59) we get

$$S_{\text{grav}}(g_0, g; m) = \frac{1}{2} \ln \frac{A}{A_0} + \frac{m^2}{4\pi} (\ln \ell - \ln 2 + \gamma + 2\pi \Psi_{\text{G}}(g_0)) (A - A_0) \\ + \frac{25}{24\pi} S_{\text{L}}(g_0, g) + \frac{m^2 A}{16\pi} \left[S_{\text{M}}(g_0, g) + 8\pi h \left(S_{\text{AY}}(g_0, \phi) - \int_{\Sigma_h} d^2x \sqrt{g^c} \phi \right) \right]. \quad (4.22)$$

As in the previous case, we may cancel the Liouville term in the gravitational action by coupling to a spectator CFT. The non-trivial part of the gravitational action is

then entirely dominated by the Mabuchi action in the case of a spherical space-time or by its generalized version including the Aubin-Yau and canonical metric terms in higher genres.

General case If we want to keep all terms of orders m^2 when $q \neq 0$, we have to deal with the functional

$$\Psi(g) = \int_{\Sigma_h \times \Sigma_h} d^2x d^2y \sqrt{g(x)} \sqrt{g(y)} R(x) G_2(x, y; g) R(y) \quad (4.23)$$

which appears in the last line of (4.19). Using the conditions

$$\Delta_x G_2(x, y; g) = -G(x, y; g), \quad (4.24)$$

$$\int_{\Sigma_h} d^2x \sqrt{g} G_2(x, y; g) = 0, \quad (4.25)$$

which follow readily from the definition (4.16), it is straightforward to express Ψ in terms of the Ricci potential,

$$\Psi(g) = - \int_{\Sigma_h} d^2x \sqrt{g} \psi^2. \quad (4.26)$$

The variation of $\Psi(g)$ and in particular its contribution

$$-\frac{q^2 m^2}{32\pi} (\Psi(g) - \Psi(g_0)) \quad (4.27)$$

to the gravitational action can then be computed straightforwardly from (3.41), but the resulting formula is not particularly illuminating. The most elegant form is obtained when $g_0 = \bar{g}$ is the metric of constant curvature, in which case (3.44) shows that (4.27) yields the contribution

$$\frac{q^2 m^2}{32\pi} \int_{\Sigma_h} d^2x \sqrt{g} \left(2\sigma + 4\pi(1-h)(\phi - S_{AY}(\bar{g}, \phi)) - \frac{1}{2} S_M(\bar{g}, g) \right)^2 \quad (4.28)$$

to the gravitational effective action.

4.1.4 The gravitational dressing

Let us introduce

$$\alpha_{q, \mathbf{k}} = \sum_i k_i - (1-h)q. \quad (4.29)$$

Let us note that in the CFT case, i.e. in the absence of mass term, the integration over the scalar field zero mode imposes the condition

$$\alpha_{q, \mathbf{k}} = 0, \quad (4.30)$$

whereas when $m \neq 0$ this parameter can be arbitrary.

The \mathbf{k} -dependent terms in (4.13), in the small mass expansion, read

$$\begin{aligned} \ln Z_{\text{mat}}(q, m; \mathbf{k}, \mathbf{x}; g) = & -\frac{2\pi(1-h)^2 q^2}{m^2 A} + \frac{2\pi\alpha_{q,\mathbf{k}}^2}{m^2 A} \\ & + 2\pi \sum_{i \neq j} k_i k_j G(x_i, x_j; g) + 2\pi \sum_i k_i^2 G_R(x_i; g) - \frac{q}{2} \sum_i k_i \psi(x_i; g) \\ & + 2\pi m^2 \sum_{i,j} k_i k_j G_2(x_i, x_j; g) - \frac{q}{2} m^2 \sum_i k_i \int_{\Sigma_h} d^2 x \sqrt{g} G_2(x_i, x; g) R(x) \\ & + \mathcal{O}(m^4) + \mathbf{k}\text{-independent terms.} \end{aligned} \quad (4.31)$$

The leading non-trivial contributions to the gravitational dressing factor (4.5) are given by the second line in (4.31). From (3.31), (3.40) and (3.41) we get

$$\begin{aligned} \ln D_{\mathcal{O}_1 \dots \mathcal{O}_n}(\mathbf{x}, \mathbf{k}, q; g_0, g) = & -\frac{2\pi(1-h)^2 q^2}{m^2} \left(\frac{1}{A} - \frac{1}{A_0} \right) + \frac{(1-h)q^2}{4} S_M(g_0, g) \\ & + \frac{2\pi\alpha_{q,\mathbf{k}}^2}{m^2} \left(\frac{1}{A} - \frac{1}{A_0} \right) + \sum_i (2 + k_i(k_i + q)) \sigma(x_i) \\ & + 2\pi\alpha_{q,\mathbf{k}} \sum_i k_i (\phi(x_i) - S_{\text{AY}}(g_0, g)) + \frac{q\alpha_{q,\mathbf{k}}}{4} S_M(g_0, g). \end{aligned} \quad (4.32)$$

The first line in the above equation cancels out with similar contributions from the gravitational effective action when computing the gravitational expectation value in (2.17). According to (2.19), the term proportional to σ in the second line provides the anomalous dimensions $\Delta(k) = -k(k + q)$ of the operators (1.8) in the massless theory. The new non-trivial part in the dressing factor is given by the terms in the third line of (4.32). Note that these terms do not contribute at $m = 0$ simply because in this case the condition (4.30) must be imposed to obtain a non-zero correlator.

Subleading terms in the gravitational dressing, which do not take a particularly simple or interesting form, can also be derived from the third line in equation (4.31). The metric variations of the relevant terms can be easily deduced from the formulas

$$G_2(x, y; g) = - \int_{\Sigma_h} d^2 z \sqrt{g(z)} G(x, z; g) G(y, z; g) \quad (4.33)$$

$$\int_{\Sigma_h} d^2 y \sqrt{g(y)} G_2(x, y; g) R(y) = - \int_{\Sigma_h} d^2 y \sqrt{g(y)} G(x, y; g) \psi(y; g) \quad (4.34)$$

together with (3.31) and (3.41).

4.2 The stress-energy tensor

The results of the previous subsection immediately yield the trace $t(x)$ of the stress-energy tensor for our model, by computing the infinitesimal variations of the effective gravitational actions,

$$\delta_g S_{\text{grav}}(g_0, g) = \frac{1}{2\pi} \int_{\Sigma_h} d^2x \sqrt{g} t \delta\sigma. \quad (4.35)$$

For example, from (3.2), (3.4) and (3.12) we find that

$$t(x; q, m) = \frac{8\pi^2(1-h)^2 q^2}{m^2 A^2} + \frac{2\pi}{A} (1 + (1-h)q^2) + \frac{25-3q^2}{12} R(x) + \frac{2\pi(1-h)q^2}{A} \psi(x) \quad (4.36)$$

for the gravitational action (4.20). The term proportional to the Ricci curvature is the standard conformal anomaly, whereas the new term proportional to the Ricci potential comes from the Mabuchi contribution in the gravitational action. Similarly, for the gravitational action (4.22), we find, using in particular (3.17),

$$t(x; m) = \frac{2\pi}{A} + \frac{25}{12} R(x) + m^2 (\ln \ell - \ln 2 + \gamma) + \frac{1}{2} m^2 (1 + \psi(x) - 8\pi h u^c(x) + 4\pi \Psi_G(g)), \quad (4.37)$$

where $u^c(x; g)$ is uniquely defined by the equations

$$\Delta u^c = \frac{1}{A} - \frac{\sqrt{g^c}}{\sqrt{g}}, \quad (4.38)$$

$$\int_{\Sigma_h} d^2x \sqrt{g} u^c = 0. \quad (4.39)$$

Using (3.10), (3.11) and (3.65), we see that

$$h u^c(x; g) = \frac{1}{2} (\Psi_G(g) - G_R(x; g)) + \frac{\psi(x; g)}{8\pi} \quad (4.40)$$

and thus

$$t(x; m) = \frac{2\pi}{A} + \frac{25}{12} R(x) + m^2 (\ln \ell - \ln 2 + \gamma) + \frac{1}{2} m^2 (1 + 4\pi G_R(x)). \quad (4.41)$$

Let us note that the constant and metric-independent terms in $t(x)$ are scheme-dependent and can be absorbed by redefining the cosmological constant. Here we have indicated the precise results in the ζ -function renormalization scheme. All the other terms are of course scheme-independent.

The trace of the stress-energy tensor could also be used to *derive* the effective gravitational action, by proceeding backward. The classical stress-energy for the action (1.7) is

$$T_{\text{cl}}^{ab} = -\frac{1}{2} \partial^a X \partial^b X + \frac{1}{4} g^{ab} \partial_c X \partial^c X + \frac{q}{2} (\nabla^a \partial^b X - g^{ab} \nabla^c \partial_c X) + \frac{m^2}{4} X^2 g^{ab}, \quad (4.42)$$

and its trace is simply

$$t_{\text{cl}} = \frac{q}{2}\Delta X + \frac{1}{2}m^2 X^2. \quad (4.43)$$

The quantum trace is then given by

$$t = \frac{q}{2}\langle\Delta X\rangle_{\text{mat},g} + \frac{1}{2}m^2\langle X^2\rangle_{\text{mat},g} + t_a(g) \quad (4.44)$$

in terms of expectation values in the matter theory and an anomalous term t_a that can be interpreted as coming from the variation of the path integral measure $\mathcal{D}X$ under Weyl transformations. From dimensional analysis, this anomaly must take the form

$$t_a(x) = aR(x) + M^2, \quad (4.45)$$

where a is a dimensionless constant and M some constant mass scale. The parameter a can be computed in the UV, i.e. in the massless scalar plus ghost CFT and it takes the familiar value $25/12$. The mass scale M can be absorbed in a redefinition of the cosmological constant and is scheme-dependent. In the ζ -function scheme, the only mass parameter in the model is m and thus we must have

$$t_a(x) = \frac{25}{12}R(x) + bm^2, \quad (4.46)$$

for some number b . We shall see below that $b = 1/2$.

The expectation values in (4.44) can be straightforwardly evaluated by computing the relevant gaussian path integrals,

$$\langle\Delta X(x)\rangle_{\text{mat},g} = -\frac{q}{2}R(x) + \frac{1}{2}qm^2 \int_{\Sigma_h} d^2y \sqrt{g(y)} G(x,y;m;g)R(y), \quad (4.47)$$

$$\langle X^2\rangle_{\text{mat},g} = 4\pi G_R^{(\zeta)}(x;m;g) + \frac{q^2}{4} \left[\int_{\Sigma_h} d^2y \sqrt{g(y)} G(x,y;m;g)R(y) \right]^2. \quad (4.48)$$

In the ζ -function renormalization scheme, we have replaced the self-contraction of X by the function $G_R^{(\zeta)}$ defined in (B.5) and related to G_R by the relation (B.6). The small mass expansions (4.14), (4.15) and equation (4.34) then yield

$$\begin{aligned} t(x) = & \frac{2\pi}{A} \left(1 + (1-h)q^2 \right) + \frac{8\pi^2(1-h)^2q^2}{m^2A^2} + bm^2 \\ & + \frac{25-3q^2}{12}R(x) + \frac{2\pi(1-h)q^2}{A}\psi(x) + 2\pi m^2 G_R^{(\zeta)}(x) \\ & + \frac{1}{8}q^2m^2\psi(x)(\psi(x)+2) - \frac{2\pi(1-h)q^2m^2}{A} \int_{\Sigma_h} d^2y \sqrt{g(y)} G(x,y)\psi(y) + \mathcal{O}(m^4). \end{aligned} \quad (4.49)$$

From this equation, we immediately find the leading non-trivial corrections to the CFT case. When $q \neq 0$, we can neglect the terms of order m^2 and we reproduce (4.36). When $q = 0$, we find a match with (4.41), taking into account (B.6) and with $b = 1/2$. In the most general case, it is straightforward to check using (3.41) that the variation

$$\frac{q^2 m^2}{32\pi} \delta \int_{\Sigma_h} d^2x \sqrt{g} \psi^2 \quad (4.50)$$

of the contribution (4.27) to the effective action precisely yields the terms in the third line of (4.49) in the stress-energy.

Of course, the same method could also be used to find again the gravitational dressing factors, in particular formula (4.32), by including into the tree-level action the terms associated with the insertion of the operators, as in (4.1). This yields the additional contribution

$$-2\pi \sum_i k_i^2 \frac{\delta(x - x_i)}{\sqrt{g}} \quad (4.51)$$

to the trace of the stress-energy and modifies in a rather straightforward way the expectation values $\langle \Delta X(x) \rangle_{\text{mat}, g}$ and $\langle X^2 \rangle_{\text{mat}, g}$. We let the details to the reader.

5 Conclusion

Studies of two-dimensional gravity have focused almost entirely on the Liouville model. This model is singled out since it universally describes the coupling of a CFT to gravity. Our aim in the present work was to motivate the study of different models, based on different actions like the Mabuchi functional, that are singled out by their nice geometrical features and their fundamental importance in Kähler geometry. Our main result was to show that these models do appear naturally in simple examples, like the massive scalar field theory studied in our work.

We believe that the study of these new quantum gravity models could be of great theoretical interest. There are many natural questions one would like to answer. For example: how does the addition of the Mabuchi action to a standard Liouville model modify the properties of the random surfaces? How does it modify the usual KPZ relation [12]? Does it help in going through the $c = 1$ barrier? What are the properties of a pure Mabuchi theory?... The new framework for the theory of two-dimensional random surfaces recently developped in [3] and [4] seems well adapted to try to address these questions, either analytically or numerically.

Many generalizations along the lines of the calculations that we have presented above can also be considered. Let us mention in particular the possibility to construct supersymmetric versions of the Mabuchi action and its generalizations by coupling to supergravity the supersymmetric version of the massive scalar field. Unravelling

the structure of these new supersymmetric actions and studying the possible supersymmetric extensions of the higher dimensional versions of the Mabuchi action as well, are examples of intriguing natural questions one would like to investigate, with potential applications both in quantum gravity and in Kähler geometry.

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A On the derivatives of the Green's function

Our aim in this Appendix is to provide an elementary derivation of (3.77). We use a coordinate system in which, locally, the metric takes the form (3.69). It is convenient to use the complex coordinates

$$u = x^1 + ix^2, \quad v = y^1 + iy^2 \quad (\text{A.1})$$

and the associated partial derivatives. We are going to show that

$$\partial_u \partial_{\bar{v}} G(x, y; g) = \frac{1}{4} \delta(x - y) - \frac{1}{4} (\text{Im } \tau)_{ij}^{-1} l_i(u) \bar{l}_j(\bar{v}). \quad (\text{A.2})$$

Since $\partial_{1a} \partial_{2a} = 2(\partial_u \partial_{\bar{v}} + \partial_{\bar{u}} \partial_v)$, the equation (3.77) used in the main text follows immediately from (A.2) in the limit $x \rightarrow y$.

The simplest way to understand (A.2) is to interpret it as a completeness relation in the space $\Omega^{(1,0)}(\Sigma_h)$ of $(1, 0)$ differential forms on Σ_h , endowed with the positive-definite hermitian scalar product

$$\langle \eta | \chi \rangle = -\frac{i}{2} \int_{\Sigma_h} \bar{\eta} \wedge \chi. \quad (\text{A.3})$$

The completeness relation follows from the the following simple special case of Hodge's decomposition theorem:

Lemma: *The space $\Omega^{(1,0)}(\Sigma_h)$ decomposes into a direct sum*

$$\Omega^{(1,0)}(\Sigma_h) = D \oplus H, \quad (\text{A.4})$$

where H is the space of holomorphic one-forms on Σ_h and D is the set of $(1, 0)$ forms which can be written as ∂f for some function f on Σ_h . Moreover, the spaces D and H are orthogonal with respect to the scalar product (A.3).

The proof goes as follows. For any $(1, 0)$ form η , $\partial\eta = 0$, which implies

$$\int_{\Sigma_h} \bar{\partial}\eta = \int_{\Sigma_h} d\eta = 0. \quad (\text{A.5})$$

By the $\partial\bar{\partial}$ -lemma we can thus write $\bar{\partial}\eta = \bar{\partial}\partial f$ for some function f on Σ_h . Indeed, in terms of components, f satisfies

$$\Delta f = -\frac{4}{\sqrt{g}}\partial_{\bar{u}}\eta_u, \quad (\text{A.6})$$

and since the integral of the right-hand side is zero this equation always have a solution for f , given explicitly in terms of the Green's function as

$$f(x) = 2i \int_{\Sigma_h} G(x, y; g) \bar{\partial}\eta(y) = -2i \int_{\Sigma_h} \bar{\partial}_2 G(x, y; g) \wedge \eta, \quad (\text{A.7})$$

where the Dolbeault operator $\bar{\partial}_2$ acts on the second argument of G . By construction, $\bar{\partial}(\eta - \partial f) = 0$ and thus

$$\eta = \partial f + \chi \quad (\text{A.8})$$

where χ is a homomorphic one-form. This is equivalent to the decomposition (A.4), the orthogonality of D and H simply coming from the fact that, if $\chi \in H$ and $\xi = \partial f \in D$,

$$\langle \chi | \xi \rangle = \frac{i}{2} \int_{\Sigma_h} d(f\bar{\chi}) = 0. \quad (\text{A.9})$$

The matrix of the scalar product (A.3) on H in the basis (λ_i) satisfying (3.60) and (3.62) is $\text{Im } \tau$ and thus the orthogonal projector Π on H is given explicitly by

$$\Pi(\eta) = (\text{Im } \tau)_{ij}^{-1} \langle \lambda_j | \eta \rangle \lambda_i. \quad (\text{A.10})$$

In the decomposition (A.8), one has $\chi = \Pi(\eta)$. Taking into account (A.7) and (A.10), Hodge's decomposition of the $(1, 0)$ -form η is thus explicitly given by

$$\eta(x) = -2i \int_{\Sigma_h} \left(\partial_1 \bar{\partial}_2 G(x, y; g) + \frac{1}{4} (\text{Im } \tau)_{ij}^{-1} \lambda_i(x) \bar{\lambda}_j(y) \right) \wedge \eta(y), \quad (\text{A.11})$$

where the Dolbeault operator ∂_1 acts on the first argument of G . This equation, being valid for any η , is equivalent to the equation (A.2), which completes the proof.

B Green's function and ζ -function regularization

The aim of this Appendix is to prove (4.18) or equivalently, using the fact that $\zeta(s)$ has a simple pole at $s = 1$ with residue $A/(4\pi)$,

$$\lim_{s \rightarrow 0} \left[\zeta(s+1) - \frac{A}{4\pi s} \right] = A \left(\Psi_G(g) + \frac{1}{2\pi} (\ln \ell - \ln 2 + \gamma) \right). \quad (\text{B.1})$$

More generally, we can give a definition of the renormalized Green's function at coincident points using the ζ function method as follows. We introduce the generalized kernel

$$\zeta(x, y; s) = \sum_{i>0} \frac{\psi_i(x)\psi_i(y)}{\lambda_i^s}, \quad (\text{B.2})$$

defined in terms of the eigenfunctions ψ_i s and eigenvalues $\lambda_i > 0$ of the laplacian. The ordinary ζ function (3.23) is simply

$$\zeta(s) = \int_{\Sigma_h} d^2x \sqrt{g} \zeta(x, x; s) \quad (\text{B.3})$$

whereas the ordinary Green's function is simply

$$G(x, y; g) = \zeta(x, y; 1). \quad (\text{B.4})$$

Subtracting the pole of $\zeta(x, x; s)$ at $s = 1$ we can define

$$G_R^{(\zeta)}(x) = \lim_{s \rightarrow 1} \left[\zeta(x, x; s) - \frac{1}{4\pi} \frac{1}{s-1} \right]. \quad (\text{B.5})$$

We are going to prove that the two definitions (3.39) and (B.5) are equivalent up to a constant shift:

$$G_R^{(\zeta)}(x) = G_R(x) + \frac{1}{2\pi} (\ln \ell - \ln 2 + \gamma). \quad (\text{B.6})$$

Equation (B.1) follows from (B.6) by integration.

The idea of the proof is to identify the part of the function $\zeta(x, y; s)$ which yields the singularity when $s \rightarrow 1$. This can be done using the integral representation

$$\zeta(x, y; s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \left(K(x, y; t) - \frac{1}{A} \right) \quad (\text{B.7})$$

in terms of the heat kernel

$$K(x, y; t) = \sum_{i \geq 0} e^{-\lambda_i t} \psi_i(x) \psi_i(y) \quad (\text{B.8})$$

and the standard small t expansion of the heat kernel,

$$K(x, y; t) = e^{-d_g^2(x, y)/(4t)} \left(\frac{1}{4\pi t} + \frac{R}{24\pi} + \mathcal{O}(t) \right). \quad (\text{B.9})$$

The singularity at $s = 1$ is due to the integration region around $t = 0$ in (B.7) and to the pole in $1/t$ in the expansion (B.9). Thus, the function defined by

$$\zeta_R(x, y; s) = \zeta(x, y; s) - \frac{1}{\Gamma(s)} \int_0^{L^2} \frac{dt}{4\pi} e^{-d_g^2(x, y)/(4t)} t^{s-2}, \quad (\text{B.10})$$

where L is an arbitrary length scale, is smooth at $s = 1$. In particular,

$$\zeta_R(x, x; 1) = \lim_{s \rightarrow 1} \zeta_R(x, x; s) = \lim_{y \rightarrow x} \zeta_R(x, y; 1). \quad (\text{B.11})$$

By evaluating

$$\zeta_R(x, x; s) = \zeta(x, x; s) - \frac{L^{2(s-1)}}{4\pi(s-1)\Gamma(s)} \quad (\text{B.12})$$

from the definition (B.10) and using $\Gamma'(1) = -\gamma$, we find

$$\lim_{s \rightarrow 1} \zeta_R(x, x; s) = G_R^{(\zeta)}(x) - \frac{1}{4\pi}(\gamma + \ln L^2). \quad (\text{B.13})$$

By similarly evaluating

$$\zeta_R(x, y; 1) = G(x, y) + \frac{1}{4\pi} \text{Ei}\left(-\frac{d_g(x, y)^2}{4L^2}\right) \quad (\text{B.14})$$

and using the expansion of the exponential integral function

$$\text{Ei}(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} dt \underset{z \rightarrow 0^-}{=} \gamma + \ln(-z) + \mathcal{O}(z^2) \quad (\text{B.15})$$

we find

$$\lim_{y \rightarrow x} \zeta_R(x, y; 1) = G_R(x) + \frac{1}{4\pi}(\gamma - 2 \ln 2 + \ln \frac{\ell^2}{L^2}). \quad (\text{B.16})$$

Equating (B.13) and (B.16) as dictated by (B.11) then yields (B.6), as was to be shown.

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